Indications of a possible symmetry and its breaking in a many-agent model obeying quantum statistics

Alexandr A. Ezhov,[1] Andrei Yu. Khrennikov,[2] and Svetlana S. Terentyeva[3]
[1]*Troitsk Institute for Innovation and Fusion Research, 142190, Troitsk, Moscow Region, Russia*
[2]*International Center for Mathematical Modelling in Physics and Cognitive Sciences, MSI, University of Växjö, S-35195, Sweden*
[3]*Moscow Engineering Physics Institute, Kashirskoe shosse 31, Moscow, Russia*

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The results of computer simulations are presented which give evidence for the existence of an interesting symmetry in a many-agent model which demonstrates, in special cases, both Bose-Einstein and Fermi-Dirac statistics. This symmetry is expressed in the close vicinity of the mean values of the degree of ultrametricity and the fraction of isosceles of the sets of agent memories (histories) coded by two different information-loss coding schemes. It is shown that this (in some sense) approximate statistical supersymmetry is probably broken at low temperatures—below some condensation limit. This breaking leads to the appearance of specific coding schemes for boson and fermion histories. The meaning of this specificity is revealed by applying the interpretation of the many-agent model described earlier [A. A. Ezhov and A. Yu. Khrennikov, Phys. Rev. E **71**, 016138 (2005)].

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I. INTRODUCTION

Many-agent models widely used in *econophysics* possess many interesting properties inherent to *complex systems* including *phase transitions* [4] with *symmetry breaking* [5]. In general, the agents considered have various *cognitive abilities* (and their limitations), e.g., different *strategies* for decision-making and *their memories* can differ in type and length (in particular, a limited memory is enough for *bounded rationality* of economical agents) [2,3]. For example, in the minority game (MG) [5], the agents memorize the number of cases where the use of a given strategy was successful, thus they were able to *count* these events. On the other hand, these agents can use a set of results of each game's turn, so they share some public accessible memory or, in general, have access to the common external information which contains the game *history*. For more complex and realistic models it is possible to suggest that sometimes the agents can extract different *incomplete* and, in a certain sense, *incompatible* information from the same sequence of events. The most challenging is not the *ad hoc* postulating of such memory codes, but the possibility to justify them, at least qualitatively, using different *symmetries* or conservation laws.

In *econophysics* many physical ideas have been used in developing many-agent models; but sometimes the relevance of, e.g., *conservation laws* is strongly criticized [6]. It is also recognized that, e.g., since in financial markets new assets appear and old ones disappear, the system is not conservative. Different *symmetries* may be used, e.g., to recognize that economical theory should produce the same results if units of currency change [7]. Besides the econophysical models can consider both homogeneous and heterogeneous ensembles of agents. For example, a phenomenon of nonequilibrium phase transition was found in a homogeneous model of negotiation dynamics [4], while many other interesting models, e.g., MG, have to be heterogeneous, i.e., consider the agents of different types. For our purpose it is expedient to consider a heterogeneous model with agents which apply different strategies and, therefore, in principle may need different schemes of memorization of their experience. One of the models with two different types of agents having two different strategies has been recently presented by authors in [1]. The agents imitating the so-called right brain and left brain strategies were found to obey Bose-Einstein and Fermi-Dirac quantum statistical distributions. Since it is known that *fermions and bosons are related in supersymmetrical theories*, it seems natural to make an effort to find also some analog of *the supersymmetry* (SUSY) in this many-agent model. This may be justified by not only abstract curiosity: as a matter of fact, we try to demonstrate that by studying this kind of symmetry one can evidence in support of reasonability to assign to the agents not only two different strategies, but also two different memories of the same sequence of events. Moreover, these different coding schemes became specific for different types of agents due to the phase transition accompanied by breaking the supersymmetry.

Remember that SUSY, a fundamental new paradigm, was discovered by Gel'fand and Likhtman [8], Raymond [9], and Neveu and Schwartz [10] in 1971. This model relates fermions (particles with a half-integer spin) to bosons (particles with an integer spin), which are the constituents of matter and forces in the standard model (SM) [11] of elementary particles. One consequence of SUSY is the existence of a *partner* for every known particle. Every fermion would have a bosonic counterpart and vice versa. In unbroken SUSY theories particles and their partners are identical in all respects except for their spins; but so far none of these partners have been observed. According to modern theory this is attributed to the fact that SUSY is a *broken symmetry*.

The supersymmetry can be considered as a symmetry under the exchange of classical and quantum physics. This is due to the fact that for the bosons which can occupy the same quantum state without any limitation there is a classical limit of the quantum system: e.g., the quantum photon field is equivalent to the classical electromagnetic field. On the contrary, there is no classical limit for fermions which can occupy only one quantum state.

Apart from applications of the SUSY idea in particle physics, quantum gravitation, etc., some efforts were made to find relations between Fermi-Dirac and Bose-Einstein distributions in statistical physics. Sometimes such relations take surprising forms: e.g., one-dimensional (1D) ideal harmonic Fermi and Bose gases have identical heat capacities [12,13].

In 1981 Witten introduced SUSY quantum mechanics [14] as an alternative to the factorization method of Schrödinger, Infeld, and Hull [15]. This mechanics is the simplest form of SUSY. The method mentioned leads to a couple of Riccati's differential equations for the fermion and boson degrees of freedom. Similar equations were also considered by Arnaud, Chusseau, and Philippe [16] and Rosu [17] where, in particular, they have shown that the action function of bosons and fermions obeys two Riccati equations similar to those obtained for superpotential arising in supersymmetric quantum mechanics. It is clear that this fact can encourage a search for SUSY properties in quantum statistics. Moreover, since the dependence of the chemical potential for bosons changes dramatically when the temperature reaches its critical value corresponding to Bose-Einstein condensation, it seems reasonable to expect the effect of SUSY breaking at this temperature. As shown below, in contrast to the particle physics where SUSY is an exact symmetry, the supersymmetry in the statistical physics can be statistical in nature too.

In this paper some numerical evidence in favor of this proposal is presented. Note that the relation of this evidence, real quantum statistics, and SUSY is not straightforward because the classical many-agent model considered below obeys real boson but only *specific* fermions statistics: the last one implies the enormous degeneration of the ground state. In a certain sense just the ensembles of bosonlike agents and corresponding ensembles of specific fermionlike agents can be treated as analogs of SUSY partners. At the same time, because of the fermion specificity considered, the symmetry between boson and fermion systems can be characterized only as SUSY-like one.

Taking account of all these circumstances we, nevertheless, believe that the SUSY argument (interpreted here as a *general symmetry between bosons and fermions*) is very useful in studying these many-agent models and this allows us to speculate that different agents can have not only different memory abilities, but are also able to apply different *incompatible coding schemes* of memories. The appearance of different specific memory schemes will be attributed to the symmetry breaking at low temperatures which, on the other hand, may be interpreted as a high level of inequality in the resource distribution in a society.

The analysis of the original many-agent model in which the agents try to hold two different resources reveals an interesting and quite reasonable *meaning of the specific codes* for bosonlike and fermionlike agents. It turns out that they are connected with the memorization of the value of the resource which is not *automatically controlled* by these agents. Note that the problem considered in this paper shares some common properties with other econophysical models, e.g., with MG it concerns the event counting memory and considers the *fluctuations* of the agent ensembles at equilibrium.

To justify all the statements given above we present some results of the computer modeling performed with a simple analog of a many-agent model introduced in [1]. This illustrates the situation where both special forms of the SUSY and SUSY breaking phenomena may be observed. The results are based on the approach initially introduced in [18] which uses *ultrametrical properties* of space for *agent histories*. In this approach two nonequivalent information-loss coding schemes are introduced and a very close vicinity of some ultrametrical properties of boson and fermion ensembles is observed at low temperatures. Here we show that a slightly modified version of the model described in [1] demonstrates an analog of approximate supersymmetry at high temperatures. This analog of supersymmetry is shown to be naturally expected at high temperatures and broken in a low-temperature region. The problem of determining critical temperature is discussed: a trivial idea that the SUSY breaking phenomenon occurs at the temperature of Bose-Einstein condensation (actually, this is quasicondensation) has not found its confirmation and, as a result, turned out to be more complicated. One of the most surprising results is that the critical temperature seems to be defined by the behavior of *fermions* rather than of bosons. In short, we suppose that the data presented in this paper give numerical evidence that the SUSY and SUSY breaking phenomena likely exist between Bose-Einstein and Fermi-Dirac ensembles.

The structure of the paper is as follows. In the Sec. II we briefly remind one of the many-agent model introduced in [1] and describe its minor symmetrical modification suggesting a symmetrical form of fermion self-interactions. In Sec. III we introduce two schemes used to code agent histories and describe ultrametric properties of their sets to be calculated and argue the possibility to observe the symmetry of these schemes for bosons and fermions at high temperatures. In Sec. IV we present the results of the study into ultrametrical properties of the sets of boson and symmetrical fermion histories in both high-temperature and low-temperature regimes. We discuss an evident symmetry demonstration in the high-temperature region and its breaking at a low-temperature limit. We note that this result is not trivial because, e.g., it has been observed in the urn model *equivalent* to the considered one but not in the *standard* urn model. In Sec. V we proceed with studying the symmetry breaking phenomenon and argue that it is probably connected with the process of fermion quasicondensation in the ground state. In Sec. VI we explain the meaning of two specific coding schemes which are used to represent boson and fermion histories. At last, in the final section we present a summary of our results and some conclusions.

II. MODELS STUDIED

Below we present the results of the study of a symmetrical variant of the *many-agent model* described in [1], its *equivalent urn model*, and a *standard urn model*; as a matter of fact, all of them demonstrate both types of quantum statistics.

The original many-agent model considers a *world* consist-

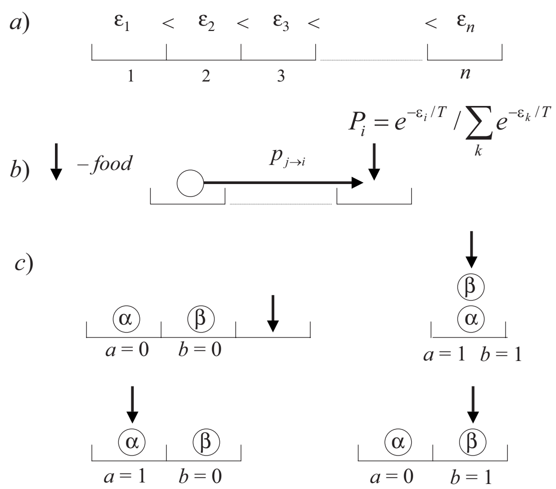

FIG. 1. (a) The model world with its cells ordered by energy values. (b) The food is presented in the cell i with a probability P_i to the agent which occupies the cell j. This agent can go to the cell i with a probability $p_{j\to i}$. (c) Four different cases of food proposals and the respective environment's proposals, a and b, to agents α and β, correspondingly.

ing of *n cells* which can be occupied by N agents. These cells are rather abstract and analogous to the energy cells, so they are not arranged in any space structure and can be ordered only by their energy values [Fig. 1(a)]. Every agent has two kinds of *resources* which have to be held positive at any time. The first resource degrades in time but can be compensated by consuming the *food* which randomly appears in the world cells from their environment. The second resource decreases each time the agent changes its cell. It is suggested in [1] that the initial amount of the second resource is high enough in order to consider its value as nonvanishing in all time intervals considered (for simplicity it may be equivalently suggested that the values of resources can be negative). The appearance of the first resource can be described by a probability f_i [Fig. 1(b)] and a cell's energy $\epsilon_i = -T \log f_i$, where the parameter T characterizes the *temperature* of the environment (see also [19]).

It is convenient to start from energy rather than from probability: in fact, we assume that the cell's energies are *equidistant*, that is typical for the energy levels of particles in a harmonic potential [20] for which Bose-Einstein condensation is subjected to both theoretical and experimental studies. So the probabilities will be calculated given the energy levels as

$$f_i = Z^{-1} \exp(-\epsilon_i/T), \tag{1}$$

where Z is the partition function.

We interpret the fact of food appearance in the cell free of a specific agent as an *environmental proposal* to enhance the first kind of resource accompanied by a decrease of the second kind of resource. Let the Boolean variable, a, denote this proposal and $a=0$ if the environment offers to change the cell. The appearance of the unit of the first resource in the cell occupied by the given agent can be considered as an offer for the agent *to preserve its second resource* and to enhance its first resource.

TABLE I. The decisions $\psi^r(a,b)$ of the right brain agent (third column) and the left brain agent, $\psi^l(a,b)$ (fourth column), which takes into account the environment proposal a to it and also the proposal b to its friend and enemy, correspondingly. The right brainer α goes to a new cell [environment proposal $a=0$ and decision $\psi^r(a,b)=0$] only if $b=1$: its friend β occupies this very cell. The left brainer α does not go to a new cell [environment proposal $a=0$ and its decision $\psi^l(a,b)=0$] only if $b=1$: its enemy β occupies this very cell.

a	b	$\psi^r(a,b)$	$\psi^l(a,b)$
0	0	1	0
0	**1**	**0**	**1**
1	0	1	1
1	1	1	1

Let $a=1$ if the environment offers the agent to keep its cell. This proposal is favorable to any agent and it is suggested that any agent will accept it *unconditionally*.

It is also assumed in [1] that every agent can *accept* (A) or *reject* (R) the proposal to change the cell ($a=0$) it occupies interacting with a *randomly chosen partner* (including itself). This situation, where all pairwise interactions are allowed, corresponds to the *mean-field topology* of the agent network [21].

Specifically, it is supposed that if food is offered to agent α, *this agent regards that it is also offered to agent* β. In accordance with Lefebvre [22], we also suppose that an agent can consider two types of relations with another agent, i.e., friendly and competitive ones. The decision of agent α depends both on the environment proposal a to agent α and also on its proposal b to agent β; *from the point of view of agent* α the same unit of the first resource is offered to agent β. Now an intention of agent α becomes a function of two variables, $\psi = \psi(a,b)$ [see Fig. 1(c)]. The intentions of these two types of agents can be considered as a function of two variables: a and b. As it is argued in [1], there are two reasonable functions of this type which can be referred to as those with *right brain dominant* and *left brain dominant* ones. In fact, for the right brain agent α which considers the *other* agent β as its friend we obtain its function values presented in the third column of Table I. The right brain agent accepts the environment proposal to consume the food in a new cell only if its friend (with which agent α interacts) already occupies it (second row of the third column). Enemies *do not influence* intentions of the right brain agent at all: this is also true for the left brain agent which interacts with a friend [corresponding values of $\psi^r(a,b)$ are not presented in Table I].

On the contrary, for the left brain agent which takes into account the situation with an enemy the decisions $\psi^l(a,b)$ are presented in the fourth column of Table I. Focusing again on the second row of this column we conclude that the left brain agent does not accept the environment proposal to consume the food in another cell, unless the environment demands of the enemy to change its cell (so that the food be offered just in the cell which is occupied by a random enemy, with which agent α interacts).

In the case of agent *self-interaction* [when it randomly chooses itself for interaction ($\beta=\alpha$)], the situation can be different. In [1] it was suggested that in this case the right brain and left brain agents act identically unconditionally accepting the environment proposal to change the original cell. This choice makes the model *asymmetrical*: the self-interacted right brain agent effectively uses the left brain strategy but not vice versa. It was demonstrated in [1] that the dynamic rules described above lead to the Bose-Einstein distribution in the population of friendly right brain agents

$$n_i(\epsilon_i) = \frac{1}{e^{(\epsilon_i-\mu)/T} - 1} \tag{2}$$

and in the population of competitive left brain agents (in the *asymmetrical model*) the equilibrium distribution is as follows:

$$n_i(\epsilon_i) = \frac{N}{e^{(\epsilon_i-\mu)/T} + 1}. \tag{3}$$

In Eqs. (2) and (3) μ denotes the chemical potential which can be determined from normalization condition

$$\sum_i^n n_i(\epsilon_i) = 1. \tag{4}$$

In this paper we concentrate on a *symmetrical* variant of the many-agent model when a *self-interacted* left brain agent effectively becomes a “right brain” one rejecting the environment proposal to leave its original cell. In this case one can easily see that at equilibrium the distribution of such agents is described by a function slightly different from that given by Eq. (3):

$$n_i(\epsilon_i) = \frac{N-1}{e^{(\epsilon_i-\mu)/T} + 1}. \tag{5}$$

Note that distributions (3) and (5) are Fermi-Dirac ones with degenerated energy levels (with the number of degenerated states $G=N$ and $G=N-1$, correspondingly).

It is easy to show that the model considered in [1] coincides with the *equivalent urn model* having the following rule of ball urn-to-urn transition: For a randomly chosen ball located in the urn j the probability to occupy the urn i chosen with probability (1) equals

$$p_{j\to i} = \frac{\mathrm{sign}(s)}{N}(s+n_i), \tag{6}$$

where n_i is the number of balls in the ith urn, and the parameter s defines the type of statistics:

(i) $s=1$ for bosons: final distribution is described by Eq. (2).

(ii) $s=-N$ for fermions with up to N balls in a cell: distribution (3).

(iii) $s=-(N-1)$ for fermions with up to $N-1$ balls in a cell: distribution (5).

Further we are to distinguish the equivalent model from the *standard urn model*, for which the transition rule is chosen as [23]

$$p_{j\to i} = \frac{(s+n_i)}{(s+n_i)+(s+n_j)}. \tag{7}$$

For the given s value they lead to the same equilibrium but the ball dynamics is different for the equivalent and standard models which display nonequivalent results in the SUSY search.

Later we show that the symmetrical many-agent model and its equivalent urn model have specific SUSY properties which are exerted in coincidence of some specific statistical characteristics of agent (or ball) histories described which have the Bose-Einstein and Fermi-Dirac distributions in thermodynamic equilibrium. Note that this symmetry manifests itself *statistically* and can be found only by averaging many statistical ensembles at equilibrium. In fact, we tend to investigate different codes of bosonic and fermionic system fluctuations in thermodynamical equilibrium.

III. CODING OF HISTORIES

A Monte Carlo modeling [24] of both the *equivalent* and *standard* urn systems can be performed according to the following procedure.

(1) Randomized initial distribution of balls in urns is generated.

(2) Random ball is chosen and its current location (urn j) is determined.

(3) Destination urn i is chosen according to probability (1).

(4) Chosen ball goes to urn i according to probability (6) or (7).

(5) Process proceeds until equilibrium state is reached and kept.

Let us investigate the history of balls in the systems at thermodynamical equilibrium. Each step of the algorithm can be characterized using the following symbolic coding. Let us call the choice of the destination urn i as a “proposal” to the ball to *go* to the urn i. The ball can either *accept* or *reject* this proposal. According to its decision the ball can either *stay* in the initial urn j or *go* to the destination urn i. Only three situations can occur (Fig. 2).

(1) The destination urn i differs from the current ball urn j and the ball *accepts* the proposal to *go* to the urn i. We can denote this situation using the pair of characters AG (*accept* and *go*).

(2) The destination urn i differs from the current ball urn j but the ball *rejects* this proposal and *stays* in the urn j. We can denote this situation using the pair of characters RS (*reject* and *stay*).

(3) The destination urn i coincides with the current ball urn j. In this case the ball unconditionally *accepts* the pro-

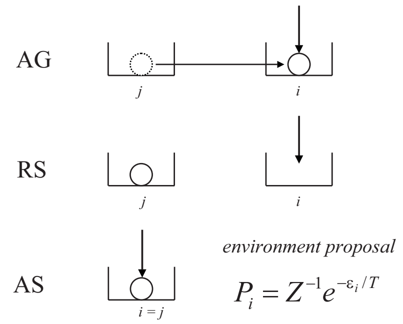

FIG. 2. Three different cases with a ball and their coding: Top: random ball accepts the environment proposal to go to the other urn; Middle: it rejects this proposal; and Bottom: it accepts the proposal to stay in the original urn.

posal and *stays* in the urn j. We can denote this situation using the pair of characters AS (*accept* and *stay*).

We can derive two special binary codes by omitting the second or the first character in the two-character code. The first choice gives us characters A and R—we name this code as AR. The second choice leaves us characters S and G and we name this code as SG [18]. If we describe the history of the given ball using the AR or SG codes, we will evidently lose some information about events. In particular, if any ball accepts the proposal (A) it can either stay in the initial urn or go to the other one. On the other hand, if the ball stays in the initial urn this implies that it either accepts the proposal to stay there or rejects the proposal to go to the other urn. Therefore it is evident that such two binary coding schemes are not equivalent: it is impossible to reconstruct the AR ball history given the SG history and vice versa. Both of these coding schemes can be considered as a projection of the full three-event history coding on two orthogonal planes. As it has been shown earlier [18], a possibility to code ball histories using two different schemes opens the way to relate bosons and fermions statistics by calculating the degree of ultrametricity of all the history sets. We demonstrate here that this permits us to study both SUSY and SUSY breaking phenomena in an equivalent urn model.

It can be expected that at high temperatures we will observe an evident symmetry between any statistical characteristics of fermion and boson history sets coded by using AR and SG (and vice versa) codes in the populations of competitive left brain agents and also cooperative right brain agents, correspondingly. Indeed, e.g., for the boson occupying the ith cell (urn) the conditional probability for acceptance (A) of the environment proposal is

$$P^b(A|\epsilon_i) = Z^{-1}e^{-\epsilon_i/T} + \sum_{j\neq i} Z^{-1}e^{-\epsilon_j/T}\frac{n_j}{N} + \sum_{j\neq i} Z^{-1}e^{-\epsilon_j/T}\frac{1}{N}$$
$$= Z^{-1}\left[e^{-\epsilon_i/T} + \frac{1}{N}\sum_{j\neq i}(n_j+1)e^{-\epsilon_j/T}\right]. \quad (8)$$

On the other hand,

$$P^b(R|\epsilon_i) = \sum_{j\neq i} Z^{-1}e^{-\epsilon_j/T}\left(1-\frac{n_j+1}{N}\right), \quad (9)$$

where

$$P^b(A|\epsilon_i) + P^b(R|\epsilon_i) = 1. \quad (10)$$

For the SG code

$$P^b(S|\epsilon_i) = Z^{-1}\left[e^{-\epsilon_i/T} + \sum_{j\neq i} e^{-\epsilon_j/T}\left(1-\frac{n_j+1}{N}\right)\right], \quad (11)$$

$$P^b(G|\epsilon_i) = \sum_{j\neq i} Z^{-1}e^{-\epsilon_j/T}\left(\frac{n_j+1}{N}\right), \quad (12)$$

where

$$P^b(S|\epsilon_i) + P^b(G|\epsilon_i) = 1. \quad (13)$$

It is easy to recognize that for fermions we have the same relations, if we replace A by S and R by G:

$$P^f(S|\epsilon_i) \equiv P^b(A|\epsilon_i), \quad (14)$$

$$P^f(G|\epsilon_i) \equiv P^b(R|\epsilon_i), \quad (15)$$

$$P^f(A|\epsilon_i) \equiv P^b(S|\epsilon_i), \quad (16)$$

$$P^f(R|\epsilon_i) \equiv P^b(G|\epsilon_i). \quad (17)$$

The probability of the symbol in a string can be obtained as

$$P^{b,f}(\{A,R,S,G\}) = P(\{A,R,S,G\}|\epsilon_i)P^{b,f}(\epsilon_i). \quad (18)$$

In a high temperature limit $P^{b,f} \propto e^{-\epsilon_i/T}$. In this case we can suggest the existence of symmetry between fermions and bosons in two binary coding schemes (AR$\leftrightarrow$SG) if we assume that the code letter appearance in the histories of different agents is not correlated. It means that if we know the characteristics of the set of boson histories in the AR coding, we immediately obtain the same values for fermions in the SG coding and vice versa. The symmetry between bosons and fermions relative to these code changes can be treated as supersymmetry. Note that *no specific codes* for fermions and bosons appear in the high-temperature region, i.e., the symmetry is reciprocal. At *low temperatures* one cannot expect that this simple symmetry is preserved because bosons and fermions are apparently subject to different statistical distributions. Nevertheless, it turns out that even in this case a remarkable relation between some characteristics of the boson and fermion history sets holds true. This is expressed as a very close vicinity of the ultrametric properties of these sets for bosons in the AR coding and for fermions in the SG coding (but not vice versa). In fact, it means there have appeared specific coding schemes for their ensembles where supersymmetry observed at high temperatures breaks. To see this phenomenon we have to consider the definiiton of the ultrametric properties just mentioned.

IV. ULTRAMETRICAL PROPERTIES OF HISTORY SETS

Ultrametricity is a well-known and widely used concept in the physics of disordered systems, especially spin glasses [25], neural networks [26], etc. As it is formulated in [27] ultrametricity implies that the distance among different states is such that they can be put in a taxonomic or genealogical tree so that the distance between two states is consistent with their position in the tree. In other words ultrametric data has a hierarchical structure. For example, in the Sherington-Kirkpatrick [28] model of spin glasses (a mean-field variant of the Edwards-Anderson model [29]) the set of ground states proves to be ultrametrical. This ultrametricity is not modified when the spin-glass disorder realization changes. It is closely connected with some other concepts, e.g., it is rooted in the *overlap equivalence* which states that all mutual information about a pair of equilibrium configurations is encoded in their mutual distance or overlap. An ultrametric topology is closely related to p-adic number theory [30]. The starting point for applying p-adic numbers to theoretical physics was a string theory—an attempt to proceed with p-adic (and more general ultrametric) amplitudes, instead of real and complex ones [31,32]. Murtagh presented many examples of using ultrametrical properties when studying practical data sets rather general in nature [33]. The goal is to find an inherent hierarchical structure in data, which can be done by calculating its *degree of ultrametricity*. This parameter can be used, e.g., for fingerprinting data sets. By doing it this way, it was found, e.g., that chaotic time series are *less ultrametric* than financial, biomedical, or meteorological time series [34].

It is also well-known [34] that high dimensional, sparsely populated data spaces can be characterized in terms of the ultrametric topology. Ultrametric spaces are characterized by a strong triangular inequality $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ for any triplet x,y,z. From this one of the most prominent features of ultrametric spaces follows that any triangle formed by any triplet is *isosceles* with two equal large sides, or is *equilateral*. It proves reasonable to study how well a set of ball histories (agent memories in many-agent modeling) can be embedded in such an ultrametric topology [34]. In this sense the question is to quantify to what degree the given metric space is ultrametric. Using two different incomplete memory codings described in the previous section we can calculate the degree of ultrametricity of the memory sets appearing in different statistical tests. We have preliminarily analyzed some agent memory sets [18] in the model described in [1] in the space of vectors with binary-valued components (corresponding to the two information incomplete schemes presented above) and the Hamming metrics. In contrast with the sophisticated approaches to the definition of the parameter proposed, e.g., in [35–37], we use just the fraction of triplets satisfying the strong triangle inequality as a measure of ultrametricity. We study one more parameter—the fraction of improper isosceles triangles to the total number of isosceles *and* proper ones—*fraction of isosceles*. In the ultrametric topology this parameter reflects a form of the data tree structure (Fig. 3). In our simulations we characterize agent's memory (ball's history) with m-component vectors $(z_1, \ldots, z_m)$, where $z_i \in \{A, R\}$ for the first coding scheme and $z_i \in \{S, G\}$ for the second one. The most surprising result of the modeling was the discovery [18] that at rather low temperatures both parameters, i.e., the degree of ultrametricity and the fraction of isosceles, seem to be very close for the memory sets of agents obeying Bose-Einstein and Fermi-Dirac statistics if the former uses the AR coding, while the latter—the SG coding. On the other hand these two parameters differ substantially for the opposite choice of coding schemes. This observation can point to a possible importance of both degree of ultrametricity and fraction of isosceles as informative parameters of particle dynamics in quantum statistics and, as it will be shown later, to a possible form of symmetry breaking.

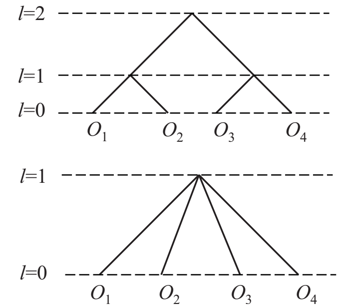

FIG. 3. The depth l of the first common ancestor of the objects O_i and O_j is used to compute the distance between them. Two trees with different degrees of isoscelity are presented. Top: all triangles for all triples (O_i, O_j, O_k) belonging to the set of four objects are isosceles [triangle sides—(1,2,2)], but not proper ones—the degree of isoscelity equals one, $I=1$. Bottom: all triangles are proper ones [triangle sides—(1,1,1)]—the degree of isoscelity equals zero, $I=0$. So the ultrametricity describes a general structure of the tree, while the fractions of isosceles describes its form.

A. High temperatures

At high temperature we really can observe the expected (rough) coincidence of two pairs of curves (for bosons in the SG coding and for fermions in the AR coding and vice versa). Figures 4 and 5 present the dependence of these two parameters for thermodynamic equilibrium ensembles of 50 agents (balls) in the symmetrical many-agent (or in equivalent urn) model obeying both Bose-Einstein statistics (2): $s=1$ in Eq. (6) and *symmetrical Fermi-Dirac distribution* (5): $s=-49$ in Eq. (6), m in interval $m \in [10, 100]$. We found that

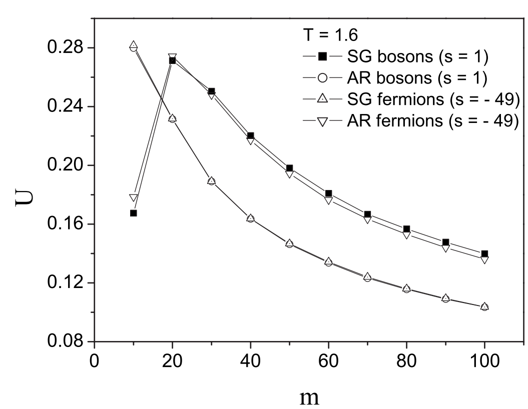

FIG. 4. For a symmetrical many-agent model (and for an equivalent urn model) at high temperatures the mean values of the *degree of ultrametricity* of the history set of cooperating right brain dominant agents (bosons) in the AR coding and ones of the competing left brain dominant agents (fermions) in the SG coding and vice versa practically coincide. $N=50$, $T=1.6$.

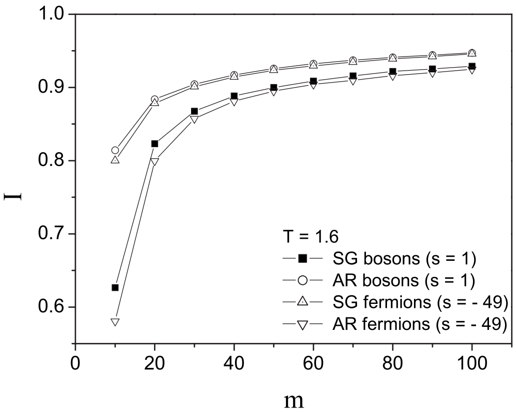

FIG. 5. For a symmetrical many-agent model (and for an equivalent urn model) at high temperatures the mean values of the *fraction of isosceles* of the history set of cooperating right brain dominant agents (bosons) in the AR coding and ones of the competing left brain dominant agents (fermions) in the SG coding and vice versa practically coincide. $N=50$, $T=1.6$.

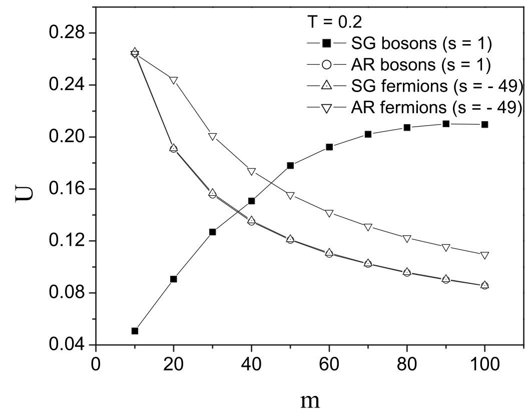

FIG. 6. For the symmetrical many-agent model at a low temperature the mean values of the *degree of ultrametricity* for boson ($s=1$) histories in the AR coding scheme practically coincide with those for fermion $[s=-(N-1)]$ histories in the SG coding, but not vice versa. $N=50$, $T=0.2$.

the system reaches equilibrium after approximately 20 updatings of each agent cell (urn) location. From Figs. 4 and 5 it follows that at $T=1.6$ average values of the degree of ultrametricity and the fraction of isosceles of the history sets of 50 agents (balls) calculated by averaging the results of 2000 tests are indeed very close for the AR coding of bosons and the SG coding of fermions [18] and vice versa in the whole interval of memory lengths.

B. Low temperatures: Appearance of specific codes

However, at low temperatures a different phenomenon is observed: mean values of the degree of ultrametricity and the fraction of isosceles for bosons in the SG coding and for symmetrical fermions in the AR coding which are very similar in a high-temperature region become entirely different. At the same time these characteristics for bosons in the AR coding and for fermions in SG coding remain very close. This means both effects of SUSY breaking and the appearance of *specific coding schemes*, i.e., AR for bosons and SG for fermions in a sense that at low temperatures their characteristics seem to be equivalent by changing the AR code for bosons to the SG code for fermions but not vice versa (Figs. 6 and 7). Note that the *closeness* of the ultrametrical parameters for complementary specific coding schemes (AR for bosons and SG for fermions) does not imply that they are *statistically indistinguishable*. On the other hand, it turns out that for many memory lengths, m, just the *fraction of isosceles* satisfies the test for the statistical equivalence (see Table II). It can indicate that this very characteristic rather than the degree of ultrametricity may be more suitable for use in the study of symmetry properties of the model.

We can admit that at high temperatures there is complete (approximate) symmetry between the coding schemes for bosons and symmetrical fermions: given the ultrametrical property value for bosons in the AR coding scheme, we can derive this value for symmetrical fermions in the SG coding scheme and vice versa. It should be noted that both the observed symmetry and its breaking for the many-agent model and for its equivalent urn model are not trivial. For example, in studying the standard urn model with transitional probability (7) we failed to find any analogical effects of symmetry and its breaking, despite the fact that both urn models have identical equilibrium distributions. So, for the symmetrical version of the many-agent model and for the equivalent urn model at high temperatures there exists an approximate symmetry among ultrametric characteristics of sets of boson and symmetrical fermions histories. On the other hand, this reciprocal coding scheme symmetry evidently breaks at low temperatures. Hence the most intriguing question is associated with the physical nature of this breaking and the determination of the phase transition temperature. In the next section we present some estimates which relate the phenomena of transition and quasicondensation of an essential part of fermions in the ground state.

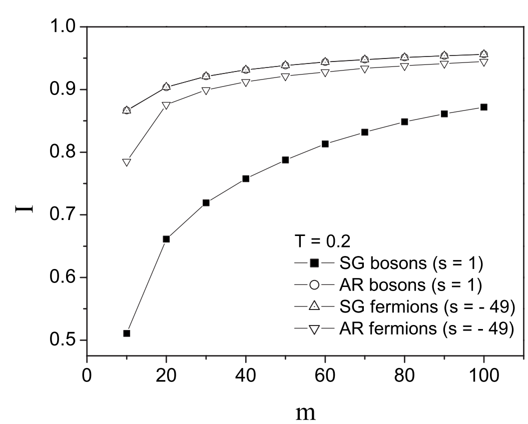

FIG. 7. For the symmetrical many-agent model at low temperature the mean values of the *fraction of isosceles* for boson ($s=1$) histories in the AR coding scheme practically coincide with those for fermion $[s=-(N-1)]$ histories in the SG coding, but not vice versa. $N=50$, $T=0.2$.

TABLE II. The degree of ultrametricity (U) and fractions of isosceles (I) of agent's histories having Bose-Einstein (2) and Fermi-Dirac (5) statistics in the AR and SG coding schemes correspondingly, at $T=0.2$ they are very close, but, in general, they are statistically different. The closeness of the corresponding values is even more profound for the fraction of isosceles, I. For some m no statistical difference of I values was revealed. The number of agents $N=50$. The mean values are obtained by averaging on 2000 trials.

m	10	20	30	40	50	60	70	80	90	100
U bosons (AR)	0.2643	0.1909	0.1555	0.1348	0.1207	0.1100	0.1022	0.0955	0.0901	0.0855
–fermions (SG)	0.2654	0.1911	0.1566	0.1356	0.1213	0.1107	0.1026	0.0959	0.0905	0.0857
I bosons (AR)	0.8666	0.9037	0.9211	0.9314	0.9385	0.9440	0.9479	0.9514	0.9539	0.9563
–fermions (SG)	0.8660	0.9039	0.9208	0.9312	0.9384	0.9437	0.9477	0.9511	0.9537	0.9561

V. STUDY OF THE SUSY BREAKING

Note that the enormous degeneration makes fermion distributions (3) and (5) even more "bosonic" than the bosonic distribution (2). It is expressed, in particular, as a sharp transition to the state where a *considerable part* of the fermions are concentrated in the ground state, i.e., sharper than in the case of quasicondensation of bosons at low temperatures (Fig. 8). This transition occurs at $T_1 \cong 0.4$. However, this is not a temperature at which all of the fermions [but one in the case of distribution (5)] concentrate in the ground state: the corresponding temperature is considerably lower ($T_2 \cong 0.02$) and at this point the chemical potential of fermions also demonstrates a sharp growth—Fig. 9. As it will be demonstrated later just the temperature $T_1 \cong 0.4$ is the critical temperature of the SUSY breaking. Another possibility that this phenomenon is connected with that of quasicondensation of bosons at higher temperatures will be rejected.

Though there is no real condensation in finite systems it is possible to identify Bose-Einstein "transition temperature" which corresponds to the concentration of a sizable fraction of N agents in the ground state. Approximately [38]

$$T_0 = \frac{N\Delta}{k \ln N}. \tag{19}$$

In our simulations ($N=50$, $\Delta=0.1$, $k=1$) $T_0 \cong 3.1$ and this value agrees with the graph of the ground state occupation by bosons (Fig. 10).

However, in our many-agent and equivalent urn models the transition takes place at a lower temperature. In Fig. 11 the distance between two curves of ultrametricity for bosons in the SG coding and for fermions (symmetrical) in the AR coding takes on small values approximately at $T \cong 0.4-0.6$. On the other hand, if we recall the dependence of the ground state occupation by bosons and fermions at equilibriums (2) and (5) (Fig. 8), we can see that this state becomes essentially occupied by fermions just at $T \cong 0.4$. The temperature of the fermion quasicondensation can be estimated from the following relationship:

$$n_0(\epsilon = 0) = \frac{N-1}{e^{(0-\mu)/T}+1} \cong \gamma N, \tag{20}$$

where $\gamma = O(1)$. From Eq. (20) it follows that

$$e^{\mu/T} = \frac{\gamma}{1-\gamma}. \tag{21}$$

On the other hand, from

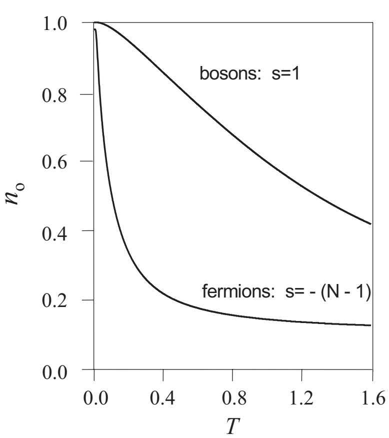

FIG. 8. Relative occupation of the ground state ($\epsilon_g=0$) versus temperature for 50 bosons and fermions distributed in ten equidistant energy states ($\Delta\epsilon=0.1$) according to forms (2) and (5). For fermions a more rapid growth occurs. These curves are calculated from the chemical potential values obtained as numerical solutions of Eq. (4).

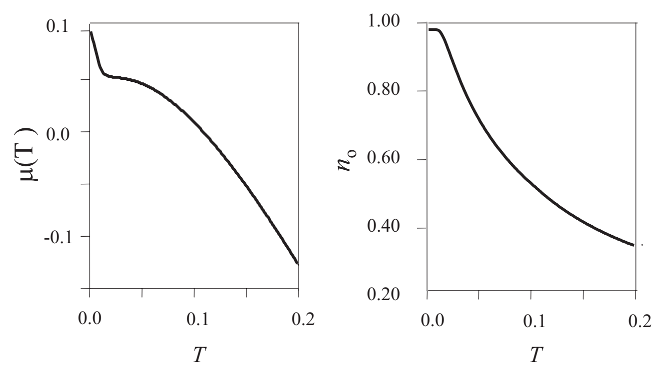

FIG. 9. The curve of chemical potential is convex at the temperatures $T > T_2 \cong 0.02$ and concave below this temperature (left). This saddle point corresponds to the condensation of $N-1=49$ fermions in the ground state which is expressed as a constant in the graph of relative ground state occupation (right).

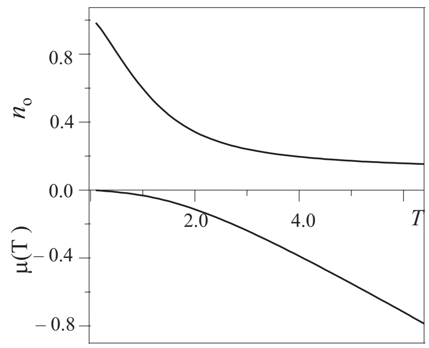

FIG. 10. Relative occupation of the ground state (corresponding to $\epsilon_g=0$) and chemical potential versus temperature for 50 bosons distributed in ten equidistant energy states ($\Delta\epsilon=0.1$). The rapid growth of the ground state occupation occurs below $T\approx 3.0$.

$$\sum_{k=0} \frac{N-1}{e^{(\epsilon_k-\mu)/T}+1} = N \tag{22}$$

and taking into account that energy levels are assumed equidistant ($\epsilon_{k+1}-\epsilon_k=\Delta$) it can be easily derived that

$$\sum_{l=1}^{\infty} (-1)^l e^{\mu l/T} Z_1(l/T) = \frac{N}{N-1}, \tag{23}$$

where

$$Z_1(l/T) = \sum_{k=0} e^{-kl\Delta/T} = \frac{1}{1-e^{-l\Delta/T}}. \tag{24}$$

For high temperatures $T \gg l\Delta$ it can be shown that

$$\frac{T}{\Delta}\log(1+e^{\mu/T}) \cong \frac{N}{N-1} \cong 1, \tag{25}$$

therefore

$$e^{\Delta/T} \cong 1+e^{\mu/T}. \tag{26}$$

From Eqs. (21) and (26) it follows that

$$T = -\frac{\Delta}{\log(1-\gamma)}. \tag{27}$$

No condensation is observed if particles are uniformly distributed in energy, i.e., $\langle N_0(\epsilon=0)\rangle \cong N/n_l$, where n_l is the number of energy levels. If the occupation of the ground state exceeds this value at least by a factor of 2, we can observe some kind of condensation. It means that

$$\langle N_0(\epsilon=0)\rangle \cong 2\frac{N}{n_l}. \tag{28}$$

In our case $N=50$, $n_l=10$, $\Delta=0.1$ we have $\langle N_0(\epsilon=0)\rangle/N \cong 0.2$, $T_c=-\Delta/\log(1-0.2) \cong 0.4$ which agrees with the temperature of fermion quasicondensation (Fig. 8).

This can mean that the breaking of the coding symmetry is presumably connected with the quasicondensation of fermions rather than of bosons, though the behavior of bosons is also relevant to this phenomenon.

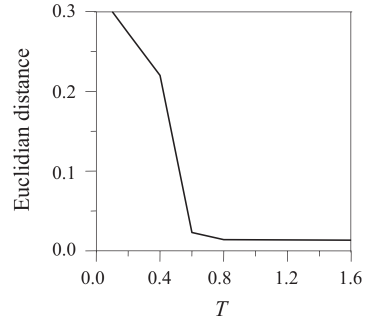

FIG. 11. The Euclidian distance between the dependencies $U(T)$ for bosons in the SG coding and *symmetrical* fermions in the AR coding shows a rapid transition at the temperature between 0.4 and 0.6.

VI. MEANING OF SPECIFIC CODING FOR BOSONS AND FERMIONS

It may be asked: what is the reason to start with a specific model of "left" and "right" brain agents [1] to demonstrate approximate SUSY properties of a more general and abstract equivalent urn model? It is surprisingly enough that a many-agent model can clarify, at least qualitatively, why the AR coding becomes specific for bosons, and the SG coding for fermions at low temperatures where the effect of SUSY breaking occurs.

In the original model the agents solve a contradictory problem: they try to hold two resources. By definition, fermions are those agents which try to accept the environment proposal (to consume food as agents), i.e., to enhance the first (*physical*) resource. On the other hand, bosons try to hold their current cell in order to save their second (so called *mental*) resource.

In this sense they have *built-in* automatic strategies which permit them to hold the first and the second resources, correspondingly. By doing so, however, they risk losing their *uncontrolled* resource (the second for fermions and the first for bosons). From this point of view for the agents it will be extremely useful to *memorize* the cases of the losing or receiving the second (for fermions) and the first (for bosons) resources, or to *count* them in order to control their quantity.

Interesting enough, for bosons this will correspond to memorizing the cases when they *accept* or *reject* an environment proposal (to consume or not consume food). On the contrary, for fermions it will correspond to memorizing the cases when they *stay* in the current cell or *go* to another one. It just corresponds to the use of the AR code in the boson memory and the SG code in the fermion memory. Such reasonability seems very intriguing in the models considered.

The case of standard urn model. It should be noted that both the observed effects of symmetry and its breaking for the original many-agent model and its equivalent urn model are not trivial. For example, when studying the standard urn model with the transitional probability (11) we found a more complicated phenomenon (Fig. 12), despite the fact that both urn models have identical equilibrium distributions. At low temperatures only the curves of ultrametric properties (especially the degree of ultrametricity) of bosons in SG coding

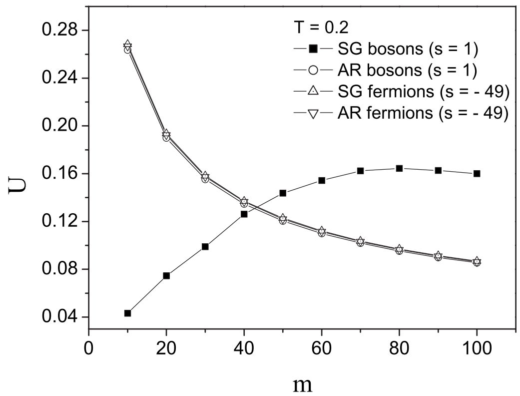

FIG. 12. The case of *the standard urn model* at low temperature $T=0.2$: no specific coding schemes are observed because, e.g., for fermions the degree of ultrametricity in the AR and SG coding schemes almost coincide both with each other and the same parameter characterizing bosons in the SG coding scheme.

change dramatically. *Three* other curves seem to be very close. So, no specific coding schemes appear. The reason is that these two forms of the urn model have evidently *different fluctuation dynamics* and, hence, different *forms of histories*.

At last, note that the symmetry appearance and breaking in the original many-agent model [1] and its equivalent urn model do not occur because of a trivial equivalence of the bosonic and fermionic models, since, otherwise, no symmetry breaking takes place.

VII. SUMMARY AND CONCLUSIONS

In summary, some extensive simulations of a symmetrical modification of the model proposed in [1] have been performed. Some arguments have been presented in favor of considering just the symmetrical variant of the original model which permits one to expect a reciprocal coding scheme symmetry (SUSY-like property) of ultrametrical characteristics of boson and fermion history sets at high temperatures. An analog of a supersymmetry operator which relates bosons and fermions can be thought of as a change of the AR coding of boson histories for the SG coding of fermions and vice versa: this converts the ultrametrical characteristics of the boson ensemble into those of the fermion ensemble (and vice versa). This operation can be performed by projection of a full three-character code onto a corresponding incomplete binary code. The symmetry studied is statistical in nature and can be seen only by averaging on many statistical realizations of agent history sets. This symmetry breaks at low temperatures, when the characteristics of bosons and fermions in SG and AR coding schemes, respectively, become entirely different (see Fig. 13). It is worth noting that these coding schemes are specific for both the fermion and boson *in-built strategies*. Indeed, the noninteracted right brain agents (bosons) stay (S) in the current cell trying to hold their second resource, while the noninteracted left brain agents (fermions) accept (A) any environment proposal. On the contrary, at low temperatures, only ultrametrical characteristics of bosons and fermions in AR and SG coding schemes, respectively, hold very similar values. These are the characteristics of agent histories, which can be memorized. This fact is interpreted as the appearance of some specific *memory codes* for both the right and left brain dominant agents in the model described in [1]. Some estimates for the critical temperature were obtained and proved to be close to that characteristic of the fermions condensation

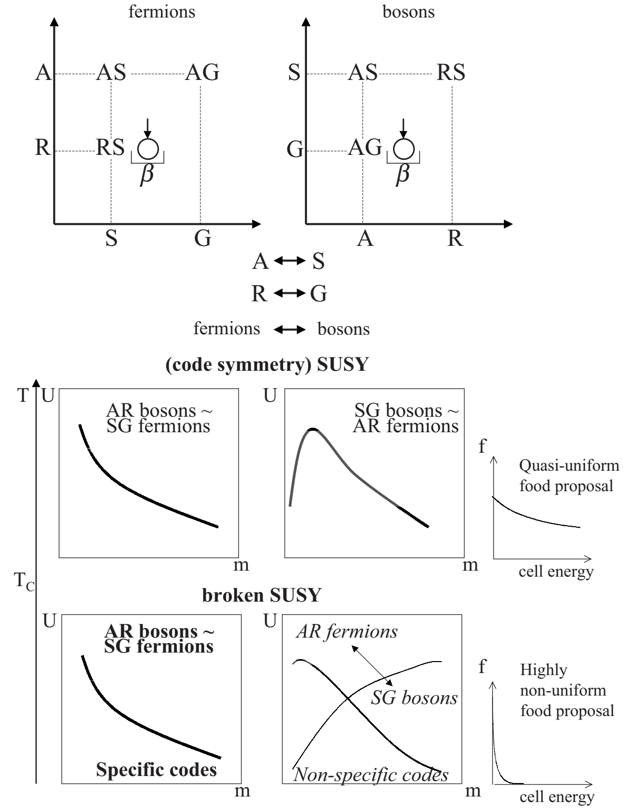

FIG. 13. Two incomplete coding schemes can be considered as projections of a full three-dimensional coding onto the two axes. For fermions (top left) the acceptance (**A**) of the environment proposal is a basic *in-built strategy* in the absence of agent interactions. The rejection (**R**) of such a proposal takes place only if food (short vertical arrow) is proposed in a cell occupied by the partner (β) the referent agent interacts with. For bosons (top right) staying in their current cell (**S**) is a basic *in-built strategy* in the absence of agent interactions. The agent goes (**G**) to the other cell only if food is proposed in a cell occupied by the partner (β) the referent agent interacts with. So the AR coding is specific in *automatic decision making* for fermionlike agents while the SG coding for bosonlike agents. The interaction rule is symmetrical under the change $A \leftrightarrow S$, $R \leftrightarrow G$, $fermions \leftrightarrow bosons$. At high temperatures (uniform food distribution) this leads to symmetry of any statistical characteristics of agent histories (middle). At low temperatures (corresponding to a highly nonuniform food proposal) this symmetry breaks (bottom). The values of ultrametrical characteristics of fermion and boson histories (agent memories) in the AR and SG coding schemes, respectively, become completely different. However, the ultrametrical characteristics of a fermion and a boson in the SG and AR coding schemes, respectively, still coincide. So the *memory-specific codes* for fermions—SG—and bosons—AR—are opposite to those relevant to *in-built* strategies.

at a highly degenerated ground state. Similar results were obtained using an equivalent urn model. By studying the standard urn model with different fluctuation dynamics, it was found that the phenomenon we are interested in was smashed. Using the many-agent formulation of the model an interesting meaning of the specific memory coding schemes for bosons and fermions is presented. This indicates that the agents of these two types have a specific memory of the events connected with the loss of the resource these agents do not control automatically. This implies that history can be *memorized* by cognitive agents and this memorizing ability becomes beneficial property needed for their survival. This also means that the left brain dominant agents (fermions) automatically enhance their first resource (if the interaction with their enemies or self-interaction does not prevent this) and memorize the second one to control its amount. In principle these agents will be able to change their strategy if the situation with the last resource becomes critical. Similarly, the right brain dominant agents (bosons) automatically hold their second resource (if the interaction with their friends or self-interaction does not enforce them to move to a new cell) and memorize the first one (food) to control its amount (again, the strategy change is possible).

In a sense, the agents considered have both a *genetic memory* (storing their in-built decision-making strategies) and a *historical memory* of their decision and action history. This makes the model more flexible and interpretive. One can say that the memory itself can be considered as a phenomenon *emerged* at the phase transition when a proposal of food becomes highly nonuniform. Note that a model which considers *mixtures of agents* of different types (right brain and left brain dominant ones) with incompatible incomplete memories is of interest in view of further studies.

Remember that the *symmetry breaking* phenomenon is an important feature of econophysical models. If these models should be *supersymmetrical*, such a phase transition could be interpreted as a sudden differentiation of the agent's memories at low temperatures which corresponds to a high inequality of environment proposals in different world cells. There are many papers which consider both *inequality* in the society [7,39] and emergent properties in complex systems [40]. From this point of view, the results of this paper support the position that economical agents themselves should be considered as complex systems with their emergent properties (specific memory) and economics as a complex system of complex systems [41].

As a matter of fact, the results presented in this paper pose more questions than they actually solve. It is unclear why the degree of ultrametricity and the fraction of isosceles are relevant properties of the SUSY and SUSY breaking phenomena in fermion and boson ensembles. We believe that in many cases we have revealed only the fact of closeness among these parameters rather than their coincidence. It is also unclear why the fraction of isosceles turns out to be a more appropriate parameter for revealing symmetry between the ensembles of boson and fermion history sets. However, it may happen that just the use of this parameter can result in a method to study ultrametrical properties of data sets of general nature [33].

Surely, it is quite interesting and important to study the effect of possible statistical indistinguishability of ultrametrical properties for different history length, m, agents number N, etc. One can also learn what happens at lower temperatures, where all fermions but one occupy their highly degenerated ground state. Note again that we have presumably outlined numerical indications to some possible symmetry which is *statistical* in nature. It may be possible to find more relevant characteristics of agent histories to express such symmetry. However, we hope that in any case it seems reasonable to continue studying the properties of history sets by applying the coding schemes discussed.

ACKNOWLEDGMENTS

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